Partial solutions to Problem Set 7

October 22, 2019

2a. Recall the Extreme Value Theorem which states that a continuous function defined on a closed and bounded set has both a maximum and minimum. Now let the set of points (living in the domain) satisfying the constraint curve be denoted S. Therefore $S = \{(x, y) | y^2 - 4x^3 + 4x^4 = 0\}$. We will invoke this theorem in the present situation by showing that S is a closed and bounded set, implying that f achieves its maximum and minimum on S.

Proof that S is bounded: We need to find a cap on how far a point in S can be from the origin. Let $(x, y) \in S$. This implies that $4x^3 - 4x^4 = y^2$. Since $y^2 \geq 0$, we have $4x^3 - 4x^4 \geq 0$. In other words, $4x^3 \geq 4x^4$.

We have 2 cases: *Case 1:* $x = 0$. Then $y^2 = 4x^3 - 4x^4 = 0$. Therefore $(x, y) = (0, 0)$. *Case 2:* $x \neq 0$. Now, if $x < 0$, then x^3 will be negative and therefore $4x^3 - 4x^4 = y^2$ will be negative, which is impossible as the square of a real number is always positive. Therefore, x is positive. Dividing our inequality $4x^3 \geq 4x^4$ by $4x^3$, a positive number (thereby retaining the direction of the inequality), we obtain $1 \geq x$. As $4x^3 - 4x^4$ is a continuous function defined on the closed and bounded interval $[0, 1]$, it achieves a maximum value (again an application of the extreme value theorem!). As a result, y^2 is bounded and therefore the y coordinate is bounded.

Since both the x and y coordinates are bounded, take the bigger of the bound and call it B. Then every point in S has a distance from the origin $= \sqrt{x^2 + y^2} \leq \sqrt{B^2 + B^2} = \sqrt{2}B$. This is the cap on the distance from the origin we were looking for, showing that S is bounded.

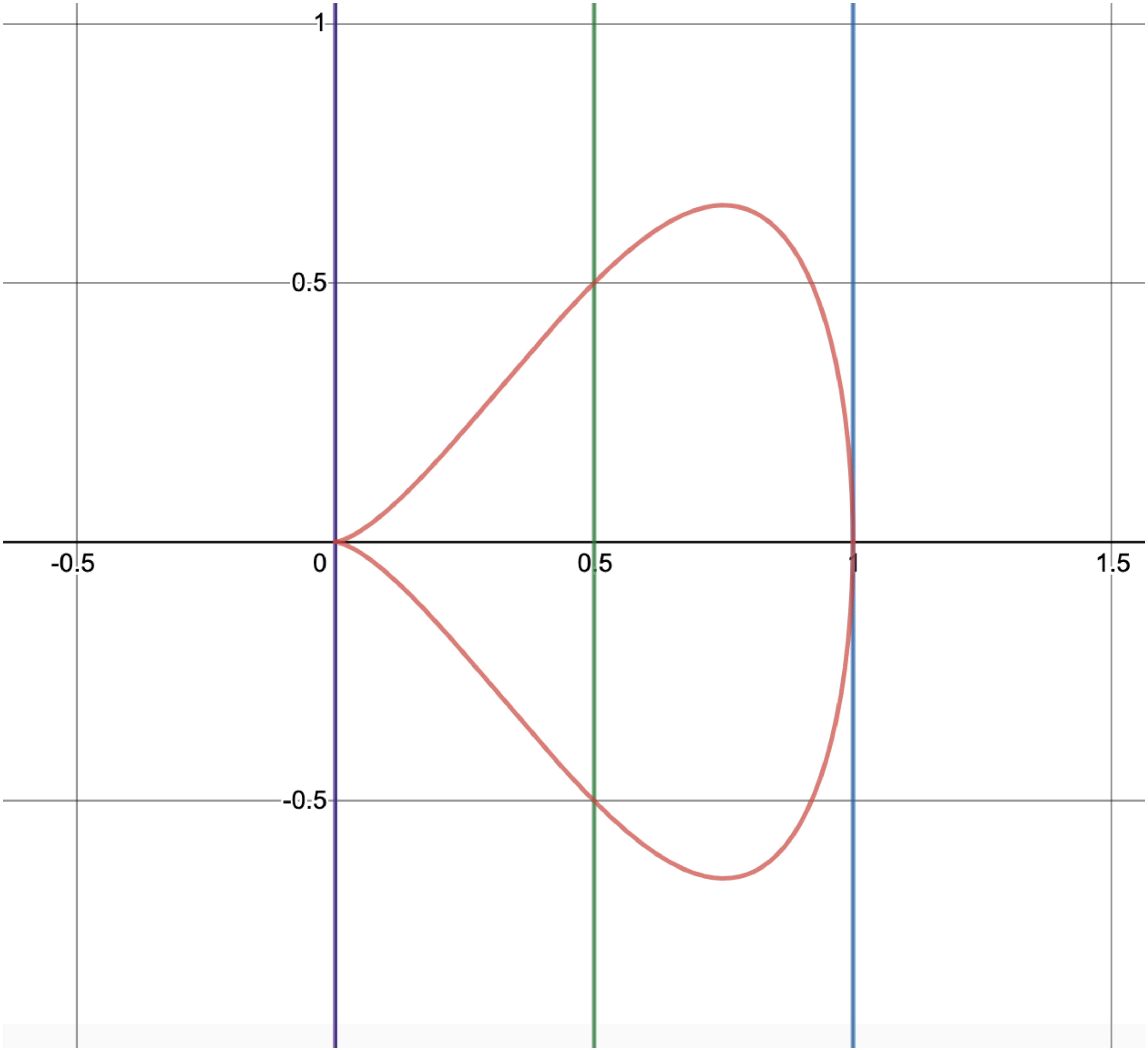
Proof that S is closed: One way to prove this is to show that the set complement of S, denoted S^c, is an open set. Recall that the complement of S is the set of all the points of $\mathbb{R}^2$ not in S. Let (x, y) be a point S^c. We want to show that there exists an open ball around (x, y) contained in S^c. Then $g(x, y) = y^2 - 4x^3 + 4x^4 \neq 0$ (by definition of S as all the points that evaluate to 0 under the function g) Let $|g(x, y)| = c$.

By continuity of g, there exists a δ such that for all points $(u,v) \in B_{(x,y)}(\delta)$, $|g(u,v) - g(x,y)| < c$.

This means exactly that $g(x,y) - c < g(u,v) < g(x,y) + c$. If $g(x,y)$ is $-c$, this forces $g(u,v) < 0$ and if $g(x,y)$ is $+c$, this forces $0 < g(u,v)$. Either way, $g(u,v) \neq 0$, and therefore $(u,v) \in S^c$. Since (u,v) is an arbitrary point of $B_{(x,y)}(\delta)$, the latter is contained in S^c. This proves that S^c is an open set and therefore S is closed.

2d. In part b, we obtain only one critical point from solving the Lagrange multiplier system, namely $(1,0)$. We obtain the other candidates for the maxima and minima by solving for $\nabla g = 0$, since the Lagrange multiplier system assumes that $\nabla g \neq 0$. The critical points obtained from the Lagrange method correspond to the points where the level set of f is tangent to the constraint curve (as these correspond to the points where infinitesimal movements along the constraint curve correspond to the f value remaining constant, as the infinitesimal movements correspond to infinitesimal movements along a level set of f. This indicates that the derivative along the constraint curve is zero and the point is critical).

So, on plotting the constraint curve along with the level sets of f, we should see that the two are tangent to each other at the point $(1,0)$. This is exactly what we observe (see figure on next page - the red curve is the constraint curve, and the blue vertical line is a level set of f corresponding to the value 1). This is seen to be the maximum because level sets for higher values of f are towards the right, and level sets for lower values of f are towards the left (the green line in the figure is the level set for the value 0.5 and the purple line is the level set for the value 0). As $(1,0)$ is the rightmost point, no higher level set of f will intersect with our constraint curve, and therefore no higher value of f will be attained. As a result $(1,0)$ is the global maximum for the restriction of f to the constraint curve. Similarly, the global minimum is the leftmost point on the constraint curve, that is $(0,0)$. Notice that for $(0,0)$, we see that the constraint curve doesn't have a smooth shape (there is a cusp) and the level set of f (the purple line) is not obviously tangent. In fact, the tangent space to the constraint curve doesn't seem to be obvious at all. (This is reasonable because $\nabla g|_{(0,0)}$ is 0. The tangent space computation for level sets of g at $(0,0)$ will show that *every* point in $\mathbb{R}^2$ is in the tangent space!) It is because of "unexpected" behavior like this that Lagrange multiplier argument doesn't work where ∇g is zero.


1
0.5
-0.5
-0.5
0
0.5
1
1.5